

Random Pinning Glass Model

Smarajit Karmakar * and Giorgio Parisi *

*Dipartimento di Fisica, Università di Roma "La Sapienza", INFN, Sezione di Roma I, IPFC CNR, Piazzale Aldo Moro 2, I-00185, Roma, Italy

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Glass transition where viscosity of liquids increases dramatically upon decrease of temperature without any major change in structural properties, remains one of the most challenging problems in condensed matter physics [1, 2] in spite of tremendous research efforts in last decades. On the other hand disordered freezing of spins in a magnetic materials with decreasing temperature, the so-called spin glass transition, is relatively better understood [3, 4]. Previously found similarity between some spin glass models with the structural glasses [5, 6, 7, 8, 9] inspired development of theories of structural glasses [10, 11, 12, 13, 14] based on the scenario of spin glass transition. This scenario though looks very appealing is still far from being well established. One of the main differences between standard spin systems to molecular systems is the absence of quenched disorder and the presence of translational invariance: it often assumed that this difference is not relevant, but this conjecture is still far from being established. The quantities, which are well defined and characterized for spin models, are not easily calculable for molecular glasses due to the lack of quenched disorder which breaks the translational invariance in the system and the characterization of the similarity between the spin and the structural glass transition remained an elusive subject still now. In this study we introduced a model structural glass with built in quenched disorder which alleviates this main difference between the spin and molecular glasses thereby helping us to compare these two systems: the possibility of producing a good thermalization at rather low temperatures is one of the advantages of this model.

Glass transition | Random Pinning | Static Correlation Length | Replica Symmetry Breaking

Dramatic slowing down of relaxation process in almost all liquids when supercooled below the melting temperature still lacks a proper explanation [1, 2]. The increase of relaxation time in deep supercooled regime is so impressive that it becomes extremely difficult for the system to reach equilibrium in experimental time scales and eventually the liquid falls out of equilibrium with further decrease of temperature to under go a calorimetric glass transition. This transition is defined at a temperature where the viscosity of the liquids becomes 10^{13} poise [18, 19]. It is clear that this transition is ad hoc in nature and depends crucially on the choice of the parameter, but the main question, which remains to be answered, is whether there is a true thermodynamic transition below this calorimetric transition temperature.

Many approaches to understand this remarkable slowing down in the dynamics of supercooled liquids invoke the existence of a cooperative length scale [20] associated with the collective rearrangements of particles and its divergence at the elusive glass transition. This scenario of glass transition is very similar in spirit to the critical phenomena seen in the continuous phase transition. The slowing down is believed to be caused by the difficulty to rearrange sets of ever increasing number of particles in a collective fashion with decreasing temperature or increasing density. Attempts to estimate this cooperative length scale remained one of the major difficulties due to the lack of identification of an order parameter characterizing the structural glass transition. This is primarily due to lack of growth of an obvious order in the system with decreasing temperature.

Progresses made in recent years to identify such a length scale is really encouraging. Dynamic heterogeneity length scale from the analysis of 4-point density-density correlation

function [22, 21, 23, 24], Point-to-set length scale [25], patch length scale [26], length scales associated with non affine displacement of particles [27], from finite size scaling of configurational entropy [24] and density of states [28] are a few to name. Unfortunately there is still no general consensus about the importance of these length scales to glass transition and their relation to each other. The main hurdle in reaching such a goal is that these length scales are only accessible in computer simulation studies because of the requirements of microscopic details to compute them. So these length scales can only be estimated in small parameter range where it grows very modestly thereby making it almost impossible to see the divergence while approaching the glass transition.

Based on these ideas of growing length scale and remarkable similarity seen in the dynamics of p -spin glassy model with the structural glasses, Kirkpatrick, Thirumalai and Wolynes [10, 13] proposed the Random First Order theory (RFOT) of glass transition. This theory is very much in spirit with the Adam-Gibbs Theory [16] proposed much earlier. This theory seems to suggest that p -spin glass models and structural glasses should belong the same universality class. However there is an inherent difference between these two models, that is the existence of quenched disorder in spin glass and not in the structural glass models. Lack of quenched disorder in structural glasses makes it rather difficult to calculate the quantities like spin-glass type order parameter and susceptibility: moreover the whole low temperature phase can not be accessed by simulations. In a recent mean field and renormalization group study [29], it was proposed that with random pinning one can explore the ideal glass phase as in the temperature - concentration of frozen particles plane there exists a critical point where relaxation time does not diverge while going from liquid to glass phase. Similar studies done in the Mode Coupling Theory frame work [30, 31] also confirm this picture. All these studies and some other recent studies [32, 33, 28, 34, 15] clearly show that exploring the glassy state in the random pinning geometry can be very insightful with the added advantage of built in quenched disorder. This can enable us to do a replica theoretic calculations for these kind of particle models to shed more light on the soundness of the replica approach to structural glasses.

In general a system of particles of equal size interacting via a radially isotropic pairwise potential will have liquid to crystal phase transition with decreasing temperature for dimensions $d \leq 3$. At higher dimensions it was shown that crystallization is strongly suppressed [35]. So to form a glass it is very important at least for dimension smaller than 4 to

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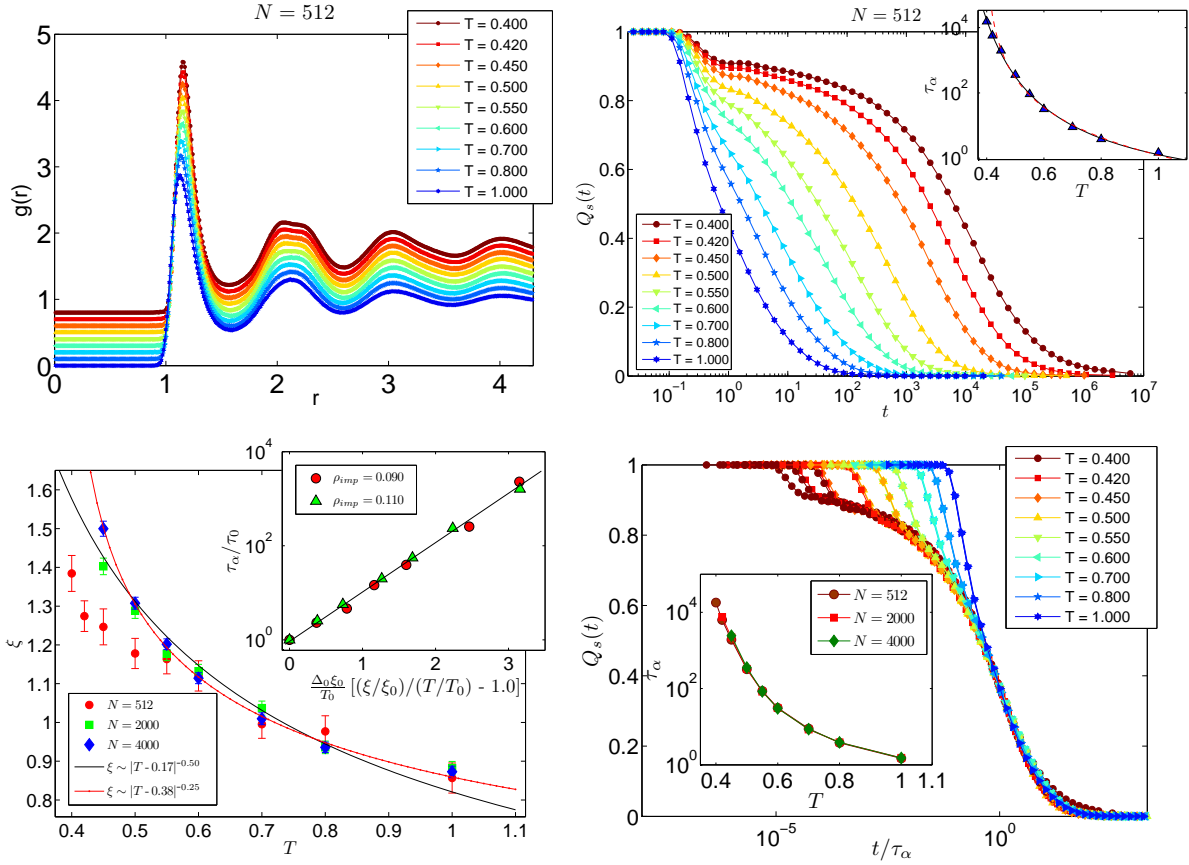


Fig. 1. Top Left Panel: Pair correlation function $g(r)$ for $N = 512$ system size for different temperatures. The curves for different temperatures are shifted vertically for clarity. Top Right Panel: Self part of the two point correlation function $Q_s(t)$ for different temperatures. The inset shows that α -relaxation time τ_α calculated as the time when $Q_s(t)$ goes to $1/e$ of its initial value. The line is the fit to the data using Vogel-Fulcher-Tamman Formula (VFT) with $T_K \simeq 0.17$. The dynamic transition temperature (MCT) is estimated to be $T_d \simeq 0.394$ (red dashed line). Bottom Left Panel: The extracted static length scale from the replica overlap correlation functions (see the method section for details). The red circles, green squares and blue diamonds are from the system with $N = 512$, $N = 2000$ and $N = 4000$ respectively. The estimation of the length scale from $N = 512$ system size seems to have some finite size effects which almost disappear for $N = 2000$ and $N = 4000$ system sizes. The black line is the fit to the data using the form $\xi(T) \sim |T - T_K|^\gamma$, with chosen $T_K \simeq 0.17$ and $\gamma \simeq 0.50$ and the red line with dot is the same fitting where we allowed all the parameters to vary and the resulting $T_K \simeq 0.38$ and $\gamma \simeq 0.25$. Inset shows the dependence of the relaxation time with the length scale for $N = 4000$ system sizes. One can see some degree of universality in the relation between relaxation time and length scale (see text for details). Bottom Right Panel: Time-Temperature superposition for the $Q_s(t)$ for all system sizes $N = 512, 2000, 4000$ for all the studied temperatures. The very nice collapse of the data confirms that these model indeed has all the usual features of a glassy system. Inset shows the temperature dependence of τ_α for different system sizes.

introduce frustration in the system to prevent it from quickly falling in to the crystalline global minimum. In spin glasses the random interactions between the spins are the source of this frustration and in structural glasses compositional disorder, for example different sizes of the particles or the asymmetric interaction between different types of particles, usually generates the required frustration. Due to presence of these extra degrees of freedom in general glass models they are often very hard to equilibrate at lower temperature as one needs also to equilibrate these extra degrees of freedom. It would be nice to have a glassy model system without extra degrees of freedom. In this article we propose to generate the required frustrations by random pinning: we have study a system where we do not have any randomness in the interaction potential nor do we have any compositional disorder. Our model system consists of particles of equal size with some fraction ρ_{imp} of them frozen randomly in space. If sufficient number of these particles is frozen randomly in space at some high temperature there is enough frustrations in the system to force the system to remain in disorder state. As there is no other degrees of freedom to equilibrate apart from the positions of the particles we expect this model can be equi-

brated to lower temperatures than the usual models to study the lower temperature phase of the supercooled liquid. With the quenched disorder, this model is also very attractive to compare it with the spin glass models. At high density of quenched particles no crystallization is present, however the glass phase is also absent: we will show that there is an intermediate region with a small, but not too small fraction of quenched particles, where a part of the usual structural glass phenomenology survives and the system does not crystallize, also at very low temperatures. Our construction differs from that of [29] as far as the quenched disorder is crucial to avoid crystallization and finding a phase transition.

Results : It turns out that one needs to freeze around $\rho_{imp} = 9\%$ of particles to get a system which will not show crystallization at least for bigger system sizes ($N > 250$). For smaller systems this amount of frozen particles is found to be not sufficient to prevent the crystallization. With $\rho_{imp} = 11\%$, even the smaller system sizes do not show any tendency to crystallize. So for the present study we choose to work with $\rho_{imp} = 11\%$ for all the system sizes studied. We restricted ourselves to small system sizes mainly because we wanted to achieve full equilibration of the system to very

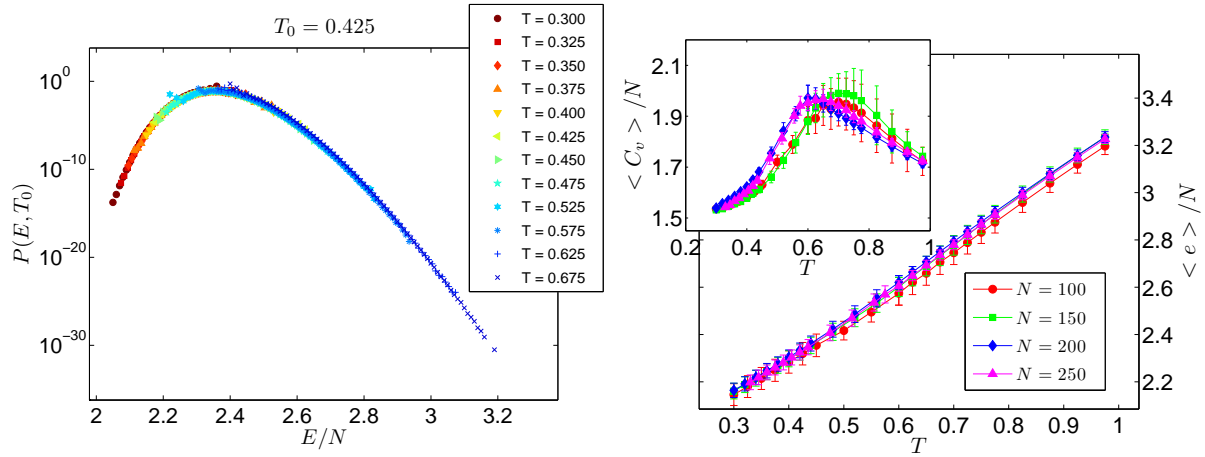


Fig. 2. Left Panel: The collapse of distribution of energy $P(E)$ for different temperature on the distribution of $P(E, T_0)$ with $T_0 = 0.425$ in the Parallel tempering run for $N = 100$ system size according to the Eq.12 to check the equilibration. The nice collapse confirms that the very good equilibration is achieved using the Parallel tempering method for lower temperatures. Right Panel: The temperature dependence of the energy for different system sizes. One can see that the finite size effect is not very strong here. Inset shows the specific heat calculated from the fluctuation of potential energy for different system size. Some finite size effect can be seen here.

low temperatures. The studied system sizes are in the range $N \in \{100, 4000\}$. In the top panel of Fig.1, we showed the pair correlation function for different temperatures to confirm that there is no sign of incipient crystallization in this temperature range. In top right panel we showed the self part of the average two point density correlation function $\langle Q_s(t) \rangle$ (see Dynamics in the Method Section for details) for $N = 512$ system size for different temperatures in the range $T \in \{0.400, 1.000\}$. One can see the nice development of the plateau in the correlation function with decreasing temperature. In the inset of this panel, we have shown the temperature dependence of the α -relaxation time and the line is the fit to the data using Vogel-Fulcher-Tamman (VFT) formula with the estimated divergence temperature $T_K \sim 0.17$ for $N = 512$ system size. The dynamic transition temperature or the mode coupling cross over temperature is estimated to be $T_d \sim 0.394$ by a power fit. One should keep in mind that these extrapolated estimation of the divergence temperatures may not be very reliable as the range of the data is not very big.

As our model is not translationally invariant due to the presence of the quenched disorder defining overlap between two replicas becomes easy and unambiguous. For example if we have a system with translational symmetry then while defining the overlap between two replicas we need to take in to account the fact that two replicas can be similar even though they may be translated or rotated in space [39]. We defined the overlap between two replicas using the window function $w(x)$ which is 1 if $x < 0.30$ and zero otherwise. We say the two replicas are similar if for each particle at position \vec{r} in replica 1 there is a particle of replica 2 within a sphere of radius 0.30 (see Replica overlap in Methods Section for details). We also defined local overlap and calculated the corresponding spatial correlation function to extract the static length scale over which the replicas are similar. In the bottom left panel of Fig.1 the extracted static correlation length of two replicas (see the Replica Overlap and Extraction of static length scale in Method section for details [37]). The finite size effect seems to die out quickly once we go to $N = 2000$ system size for lower temperatures. The modest growth of the static length scale in this temperature range is very similar to other glass models. The inset shows the dependence of the relaxation time with this length scale (see the relation between length scale and

relaxation time in Method section for details). The apparent universality for two different frozen particles density is also in agreement with the recent findings for usual glass models [36, 38].

To achieve equilibration at further lower temperature we implemented Parallel Tempering simulation methods following [40], but restricted ourselves to systems up to $N = 250$. The details of the simulation method and the parameters used are given in Parallel Tempering methods parts in Method Sections. We parallelized the Parallel Tempering method using Message Passing Interface (MPI) routines to speed up the simulation. We run different replica in different computer cores and we found that the system can be equilibrated to very low temperature within reasonable CPU time (around 12 hours for $N = 250$ particles using 16 replicas in 16 cores). We checked that system does not crystallize even at the lowest temperature studied. We believe parallel tempering methods works so well for our model as it has only positions to equilibrate and does not have any other compositional disorder which needs further equilibration. In left panel of Fig.2, we have collapsed the probability distribution of potential energy $P(E, T)$ for different temperatures according to the ansatz Eq.12 on the probability distribution of the reference temperature $T_0 = 0.425$, to ascertain whether proper equilibration is achieved in our simulations (see Parallel tempering part in Method section for details). The nice collapse of the data indicates that the equilibration is achieved to a very good accuracy. The right panel of Fig.2 shows the temperature dependence of the average potential energy and the inset shows the corresponding specific heat calculated from the fluctuation of potential energy. Although average potential energy does not show strong finite size effects one can see somewhat strong finite size effect in the specific heat. The nicely developed peak in the specific heat seems to be a precursor to a possible second order phase transition as also seen in [17]. The usual discontinuity in the specific heat seems to remain rounded in a volume independent fashion. Notice that at the lowest temperatures we have simulated the specific heat drops to the DulongPetit value, suggesting that harmonic degrees of freedom are the mostly relevant ones at these temperatures.

In Fig. 3, the temperature evolution of the distribution of overlap q is shown for six different system sizes. One can clearly see the change of the distribution from Gaussian to Bi-

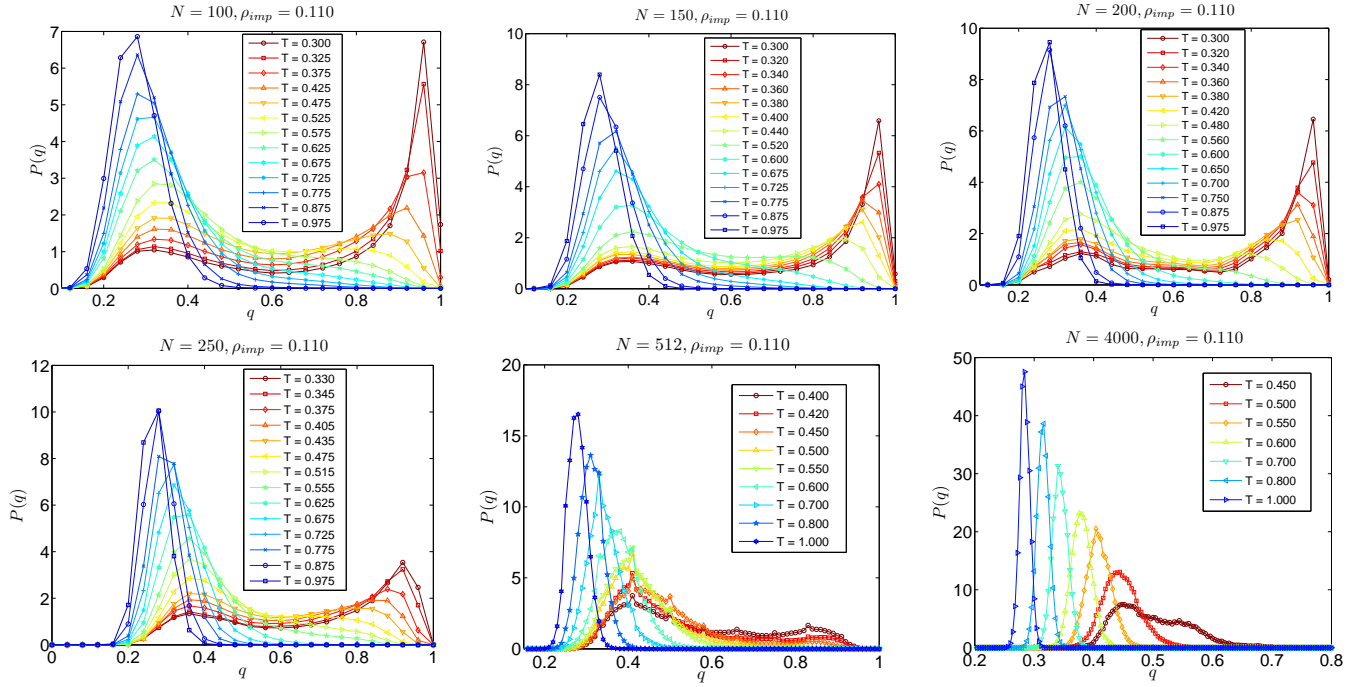


Fig. 3. Temperature evolution of the Probability distribution of overlap q for six different system sizes $N = 100, 150, 200, 250, 512, 4000$.

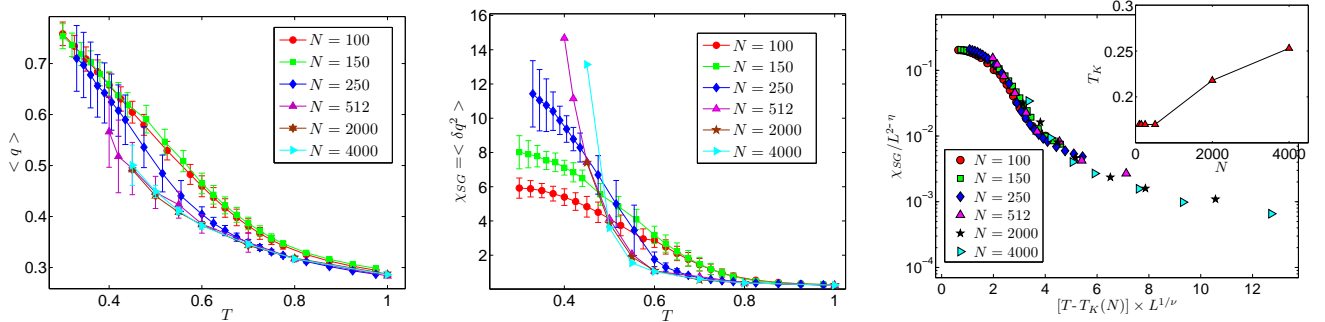


Fig. 4. Left Panel : Temperature dependence of overlap for different system sizes. Bigger system size seems to show sharper change of overlap q with temperature. Middle Panel: The spin glass susceptibility is plotted as a function of temperature for the studied system sizes. The strong variation with temperature is really remarkable. Notice the strong system size dependence of susceptibility at lower temperature. Currently we can not say whether this susceptibility will diverge with decreasing temperature in thermodynamic limit as the system sizes studied here are still quite small. One needs to do bigger system sizes to find out the possible divergence of the spin glass susceptibility at lower temperature. Right Panel: Scaling collapse of the spin glass susceptibility using the scaling ansatz Eq.4. The exponents are $\eta \simeq -0.1$, $\nu \simeq 1.0$ with T_K as shown in the inset.

modal with decreasing temperature. The distribution seems to deviate from the Gaussian one at temperature close to the temperature where the specific heat also shows peak as a function of temperature for that system size. The shape of the distribution as function of the temperature recalls what happens in mean field theory in the replica approach: below the critical temperature a peak at higher values of q appears while the low q peak has an intensity that is proportional to the temperature.

In left panel of Fig. 4, the temperature dependence of the overlap (see Method section for detail definition) for different system sizes are shown and one can clearly see that for larger system sizes the overlap seems to change more sharply. The corresponding susceptibility is given by $\chi_{SG} = (N - N_{imp})[\langle \delta q^2 \rangle]$, where $\langle . \rangle$ is the thermal averaging

and $[.]$ means the averaging over the different realizations of the disorder. N_{imp} is the number of impurity particles in the system which is given by $N_{imp} = \rho_{imp}N$. In middle panel of Fig.4, the susceptibility shows dramatic variation with temperature and at lower temperature the susceptibility seems to change quite strongly with system size also. For system size $N = 250$ the susceptibility changes almost by a factor of 50 compare to its high temperature value. With this data divergence of the susceptibility at lower temperature in the thermodynamic limit can not be established but the strong increase of its value with the system size is encouraging. we have tried to collapse the susceptibility data for different system sizes using the finite size scaling ansatz ($L \equiv N^{1/3}$):

$$\chi_{SG} = L^{2-\eta} \mathcal{F} \left(L^{1/\nu} |T - T_K| \right), \quad [1]$$

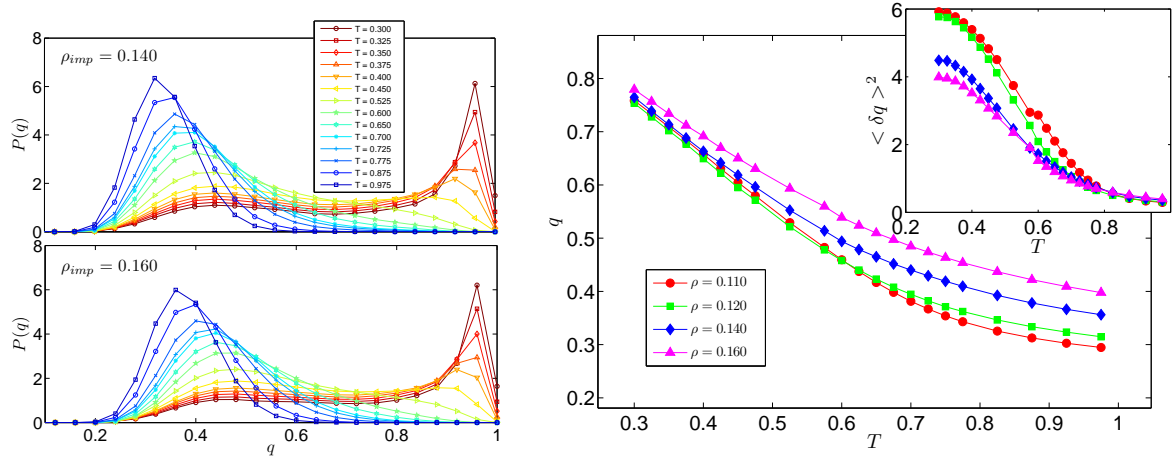


Fig. 5. Data for $N = 100$: Left Panel : The distribution of overlap $P(q)$ system size for two different concentration of frozen particles. There is not a strong qualitative difference between these distributions for two concentrations. Right Panel : The average overlap $\langle q \rangle$ as a function of temperature for different concentration of frozen particles. Top right corner inset shows the dependence of susceptibility with concentration ρ_{imp} .

with rather poor results. A better collapse is shown in the right panel of Fig. 4 with $\eta \simeq -0.1$, $\nu \simeq 1.0$ and $T_K(L)$ now being a function of system size L . It is important to notice that as T_K grows with system size it is reasonable that the divergence temperature will not go to zero in the thermodynamic limit. The data seems to indicate a possible non-zero temperature thermodynamical transition of second order in nature. The value of ν is somewhat larger than the exponent $\nu = 2/d$ argued in RFOT theory [10, 13] by scaling arguments. These results are very similar to the one obtained for finite range p -spin glass model in 3 dimension in [17]. The inability to collapse the data with a size independent critical temperature may be the effect of the strong finite size scaling corrections: the system sizes studied here are relatively small. A different scenario would be (in the thermodynamic limit) a strong increase of the susceptibility approaching the mode coupling transition temperature followed by a cross over to a different behavior at low temperature. We noted that the susceptibility data for $N = 4000$, $T \geq 0.45$ (that should have small finite size effects) are well fitted by a power law, with a critical temperature around 0.4, i.e. the putative mode coupling transition. Much more extensive simulations are needed to better understand the nature of the transition, if any.

Mode Coupling Theory calculation in similar geometry for the hard sphere system in [31] suggests that at some critical density ($\rho_{imp} \sim 0.15$ in [31]) of the frozen particles the time dependence of the two point correlation function will change from being two steps to one step. So to check whether we are not very close to this critical density we performed similar studies for different ρ_{imp} for $N = 100$ and calculated the overlap distribution $P(q)$. One can clearly see in Fig.5, that there is no qualitative change in the behaviour with different pinning density. In right panel we show how $\langle q \rangle$ changes with ρ_{imp} as a function of temperature and the top right corner inset show the corresponding susceptibility.

Conclusions:

To conclude we showed how a very simple particle model with random pinning can be used to explore the glassy phase at deep supercooled regime and also showed how this model can be used to compute different spin glass correlation functions to shed light on the relation between spin glass transition and structural glass transition which is the basis for most of the recent theories of structural glasses. This model can also be used to find out the relation between different length scales

as in this model the length scale is calculated directly from the spin glass order parameter correlation function. It would be extremely interesting to arrive to a precise determination of the phase structure of the model and to compare it with accurate numerical simulations.

Materials and Methods

Simulation details : The interaction potential for $r_{ij} \leq x_c$ is given by

$$\phi(r_{ij}) = \frac{k}{2} \left[(r_{ij})^{-k} + \sum_{\ell=0}^q c_{2\ell} (r_{ij})^{2\ell} \right] \quad [2]$$

while it is 0 for $r_{ij} > x_c$, where r_{ij} is the distance between particle i and j , and x_c is the dimensionless length for which the potential will vanish continuously up to q derivatives. The coefficients $c_{2\ell}$ are given in [28].

Dynamics: We studied the dynamics using two point overlap correlation function, $Q(t)$, defined as

$$Q(t) = \int d\vec{r} \rho(\vec{r}, t_0) \rho(\vec{r}, t + t_0) \sim \sum_{i,j=1}^N w(|\vec{r}_i(t_0) - \vec{r}_j(t_0 + t)|) \quad [3]$$

where $\rho(\vec{r}, t)$ is particle density at space point \vec{r} at time t , and $w(r) = 1$, if $r \leq a$ and zero otherwise. Average over the time origin t_0 is assumed. The use of the window function [$a = 0.30$] is to remove the fluctuations in particle positions due to small amplitude vibrational motion. In Eq.3 the contribution only due to the self-term is denoted as $Q_s(t)$. The structural relaxation time τ_α is the time where $Q_s(\tau_\alpha) = 1/e$.

Replica overlap and Extraction of static length scale: In this section we will explain how we have extracted the growing static length scale in this system. We followed the method used in [22]. The method is explained briefly below, we start with the following definition of overlap between two replica

$$q = \frac{1}{N} \sum_{i=1}^N q_i \text{ where } q_i = \sum_{k=i}^N w(x_i - x_k), \quad [4]$$

with $w(x) = 1$ for $x < 0.3$ else 0. Now we define the coarse grain variable as

$$\mu(x) = \sum_i \delta(x - x_i) q_i, \text{ and defined } f(r) = \langle \mu(x) \mu(y) \rangle \quad [5]$$

where $r = |x - y|$. Now to remove the natural oscillation in the function $f(r)$ we divide the function by pair correlation function $g(r)$ to define another function $c(r) = f(r)/g(r)$, and fitted the function to the following fitting function

$$\tilde{c}(x) = a + \exp(-x/\xi) [b + c \cos(xd + e)] \quad [6]$$

to extract the length scale ξ . We extracted the correlation length scale for the $N = 512, 2000$ and 4000 system sizes which have been equilibrated using standard molecular dynamics simulations using Berendsen thermostat [41] and we studied the system in the temperature window $T \in \{0.400, 1.000\}$. We averaged the data over 20 different realizations of the disorder. For last three temperatures $T = 0.400, 0.420$ and 0.450 , we averaged the data over 40 realizations of the disorder.

Relation between length scale and relaxation time : We start with the ansatz for the relation between relaxation time τ_α and the static correlation length ξ .

$$\tau_\alpha(T) \propto \exp\left[\frac{\Delta_0 \xi(T)^\psi}{T}\right], \quad [7]$$

where Δ_0 is a non-universal coefficient that depends on the details of the glass former. Now at reference temperature $T = T_0$ (the highest temperature in this case) we define the typical length to be ξ_0 . Then the relaxation time at that temperature is

$$\tau_\alpha(T_0) \propto \exp\left[\frac{\Delta_0}{T_0}\right], \quad [8]$$

So we have the following relation

$$\log\left[\frac{\tau_\alpha(T)}{\tau_\alpha(T_0)}\right] = \frac{\Delta_0 \xi^\psi}{T} - \frac{\Delta_0 \xi_0^\psi}{T_0} = \frac{\Delta_0 \xi_0^\psi}{T_0} \left[\left(\frac{\xi}{\xi_0}\right)^\psi - 1\right]. \quad [9]$$

As the pre-factor Δ_0 is not known a-priori for different models we choose $\Delta_0 = 1.0, 1.11$ for $\rho_{imp} = 0.090$ and 0.110 respectively in Fig.1. We also choose $\psi = 1.0$ for these two cases.

Parallel tempering methods : To equilibrate the system still at lower temperature, we have implemented parallel tempering method. We briefly mention the method here as details can be found in [40]. We construct a system consisting of M non-interacting subsystems (replicas), each consists of N particles with phase space coordinate $\{P^i, Q^i\}$, where $P^i = \{p_1, p_2 \dots p_N\}$ and $Q^i = \{q_1, q_2, \dots q_N\}$ for the i^{th} subsystem. The Hamiltonian of the i^{th} subsystem is given by

$$H_i(P^i, Q^i) = K(P^i) + \Lambda_i E(Q^i), \quad [10]$$

where K is the kinetic energy, E is the potential energy of the system and $\Lambda_i \in \{\lambda_1, \dots \lambda_M\}$ is the parameter to rescale the potential energy. Now we perform molecular dynamics simulation of the whole system with Hamiltonian $\mathcal{H} = \sum_i^M H_i$ at a reduced temperature $T = 1/\beta_0$ using a modified isokinetic simulation method. Time step for the MD is taken to be $\delta t = 0.005$. Now at each time interval of $\Delta t_X = 1000\delta t$ we exchange the parameter λ between different replica i and j keeping all other things unchanged. The exchange is accepted using a Metropolis scheme, with a probability

$$w_{i,j} = \min(1, \exp(-\Delta_{i,j})) \quad [11]$$

where $\Delta_{i,j} = \beta_0(\Lambda_j - \Lambda_i)(E(Q^i) - E(Q^j))$. We perform these steps for sufficiently long time such that we get proper equilibration of the system. This way we generate canonical distribution at inverse temperatures $\beta_i = \lambda_i \beta_0$. Here we have also parallelized the code using MPI to speed up the simulation process. We have used $M = 12$ replicas between temperature 0.300 to 0.600 and another 12 replicas for temperature range 0.600 to 1.000 for $N = 100$ and 150 system size. For $N = 200$, we have used 16 replicas for 0.300 to 0.600 temperature range and 12 between 0.600 to 1.000 . For $N = 250$, we only tried to equilibrate the system up to $T = 0.330$ and used 16 replicas between temperature range 0.330 to 0.600 and 12 between 0.600 to 1.000 . We have averaged the data over 400 different realizations of the disorder for systems $N = 100, 150, 200$ and 250 . So the estimated computer time for $N = 250$ system size is close to 10^5 hours. To check the equilibration of the system we used the method as in [40] to rescale the Canonical distribution function using the formula

$$P_i(E; T_j = \lambda_j \beta_0) = \frac{P_i(E) \exp[(\lambda_i - \lambda_j) \beta_0 E]}{\int dE' P_i(E') \exp[(\lambda_i - \lambda_j) \beta_0 E']}, \quad [12]$$

In equilibrium the left hand side of the above equation should be independent of i to within the accuracy of the data as can be seen in left panel of Fig.2.

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Supplementary Material : Random Pinning Glass Model

Smarajit Karmakar and Giorgio Parisi

*Departimento di Fisica, Università di Roma "La Sapienza", INFN,
Sezione di Roma I, IPFC CNR, Piazzale Aldo Moro 2, I-00185, Roma, Italy*

I. TIME-TEMPERATURE SUPERPOSITION FOR $Q(t)$:

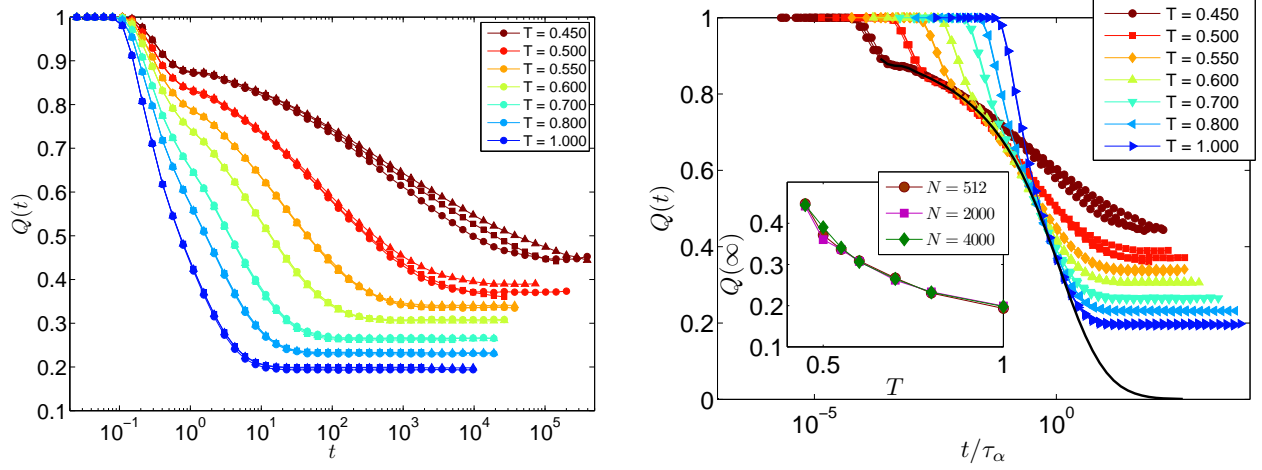


FIG. 1: Data for $N = 512$, $N = 2000$ and $N = 4000$. Left Panel: System size dependence of the relaxation function $Q(t)$. One can see that there is very small finite size effect in these system sizes. Right Panel: The time-temperature superposition of $Q(t)$ using the relaxation time τ_α obtained from the self part of $Q(t)$ from the condition $Q_s(\tau_\alpha) = 1/e$. It clearly shows that the initial part of the relaxation function is very much same as that of the self part but deviates from it as the infinite time value of $Q(t)$ is not 0 in this case. The black solid line is the master curve one would obtain if one tries to do the time-temperature superposition for $Q_s(t)$ as done in Fig.[1] in main article. This is shown just to point out that the master curve for $Q(t)$ also follows the same curve but deviates from it due to non-zero asymptotic value. Inset shows the temperature dependence of the infinite time value of $Q(t)$ that is $Q(\infty)$ for all the system sizes. One can clearly see that there is hardly any system size dependence in this quantity.

In SMFig.1, we have shown the time-temperature superposition for the full overlap correlation function $Q(t)$ with the same relaxation time τ_α which are used to collapse the data for self part of the correlation function, i.e. $Q_s(t)$. The figure clearly shows that initial decay of the correlation function is same as that of the self part only. $Q(t)$ deviates from the master curve because of the non zero asymptotic value $Q(\infty)$. In this figure we also plotted the data for all system sizes. Different color indicates the temperatures and symbols distinguishes the different system sizes. The inset shows the temperature dependence of the asymptotic value of $Q(t)$ i.e. $Q(\infty)$ and one can clearly see that there is very small finite size effects in this quantity.

II. EXTRACTION OF STATIC LENGTH SCALE FROM THE REPLICA CORRELATION FUNCTION :

As discussed in the method section of the main paper about how we extracted the length scale scale from the replica correlation function, we have presented here some data for the $N = 4000$ system size in SMFig.2. We used the method explained in the Replica Overlap and Extraction of static length scale section of the Method section of the main paper. The nice fitting of the data over the whole range and for all temperatures indicates that the extracted length scale is very reliable. This can also be seen from the data shown in Fig.[1] of the main paper where we compared the results of this fitting for different system sizes. The data for $N = 2000$ is very close to the one obtained from $N = 4000$ system size. This implies that this method of extraction of the length scale is very robust once we rule out the finite size effect in the data.

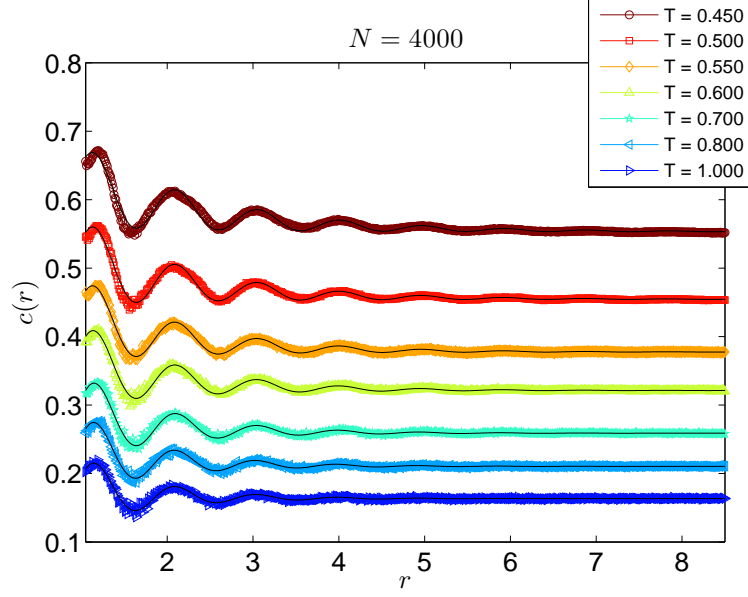


FIG. 2: Extraction of the static length scale for $N = 4000$ system size. We used the functional form $\tilde{c}(x) = a + \exp(-x/\xi) [b + c \cos(xd + e)]$ to fit the data. The nice fitting over the whole range of the data indicates that the length scale extracted this way is very reliable. Data for different curve is shifted vertically by 0.01 from each other for clarity.

III. CHECK OF EQUILIBRATION IN PARALLEL TEMPERING RUNS:

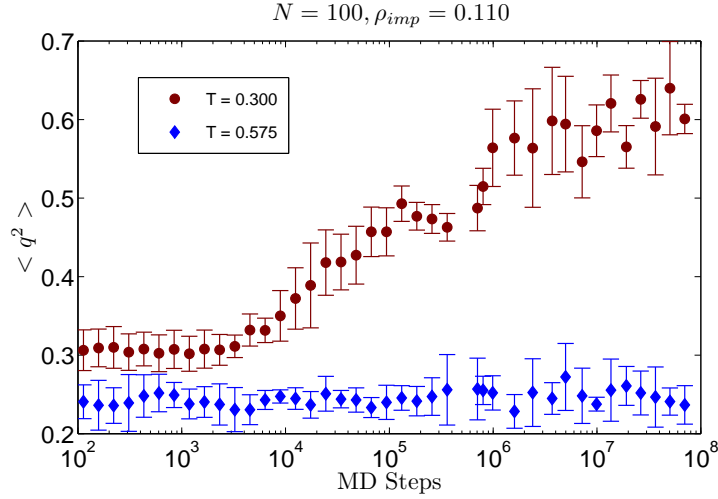


FIG. 3: Evolution of $\langle q^2 \rangle$ as a function of MD steps to check whether the time evolution of $\langle q^2 \rangle$ reached a plateau. We take the sample to be equilibrated if $\langle q^2 \rangle$ reaches a plateau value within the error bar.

We performed two separate tests to check the equilibration of our sample : one is on the distribution of energy $P(E)$ where we tried to rescale all the $P(E)$ for different temperatures to collapse on the distribution of our reference temperature T_0 as shown in the Fig.[2] of the main article using the ansatz Eq.[12] in Parallel tempering part of the Method section. Here we present the data for the second test. The test consists of checking when square of the replica overlap $\langle q^2 \rangle$ reaches a steady plateau value with time. In SMFig.3, we showed data for $N = 100$ system size and the data clearly shows that our simulation time is longer than the time required to reach a plateau value for $\langle q^2 \rangle$. As we have started from two completely uncorrelated replicas their initial overlap is very small and over the time it

reaches the equilibrium value from below.

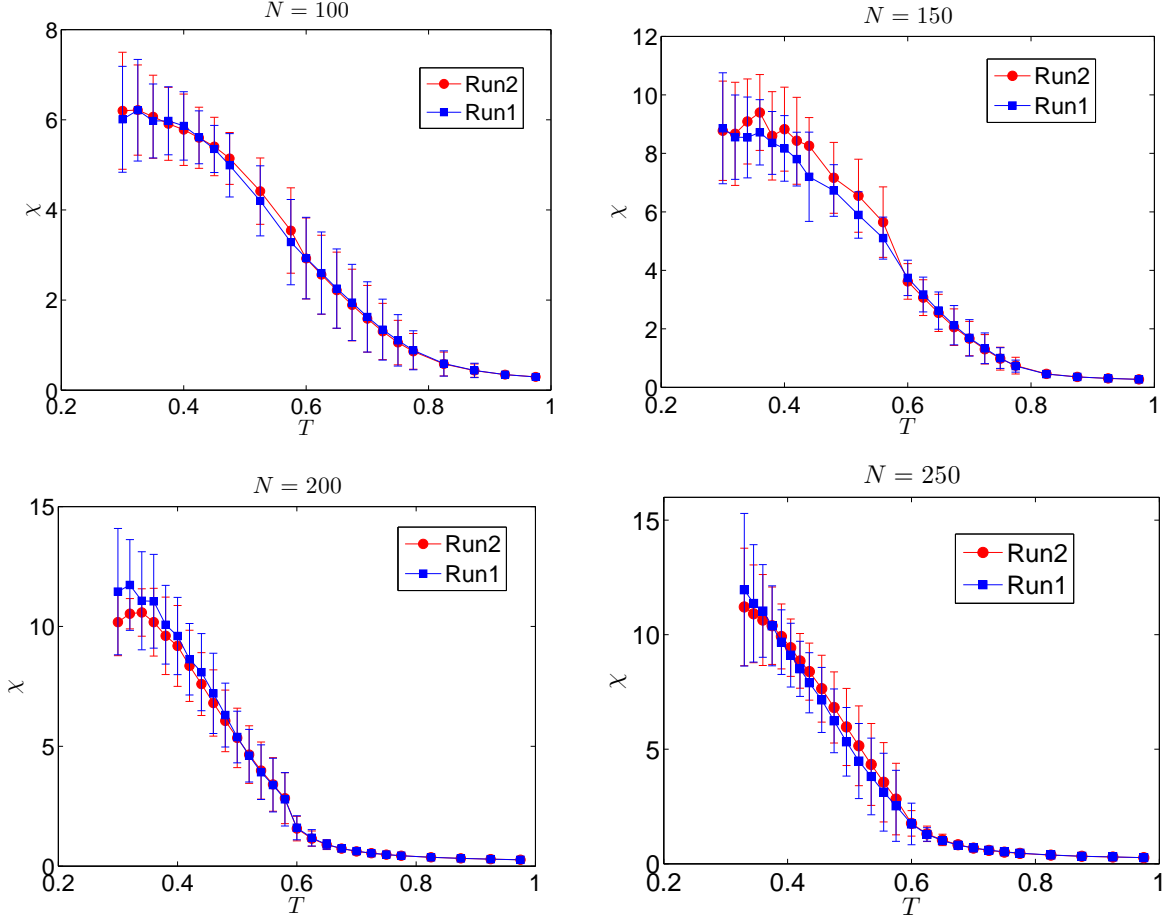


FIG. 4: Comparison of susceptibility χ for two sets of runs for different system sizes. Run2 is 2 times longer than Run1. Error bars are calculated by bootstrap method.

In SMFig.4, we have shown the spin glass susceptibility χ calculated for all the 4 different system sizes for two different run length. Run1 is 1×10^8 MD steps long and the Run2 is 2 times longer than the Run1. The value of the susceptibility is within error bar of each other. This also confirms that our data is well equilibrated.

IV. SAMPLE TO SAMPLE VARIATION OF $P(q)$:

In this section we have shown the sample to sample fluctuations of the distribution of $P(q)$ for two different system sizes at the lowest temperature simulated for two system sizes $N = 150$ and 250 . The SMFig. 5 shows $P(q)$ for 8 randomly chosen different samples. One can clearly see the strong sample to sample fluctuations of this distribution which is also very similar to spin glass models [1]. Some samples have only two peaks, with a deep valley in between, while other samples have a third peak.

V. SKEWNESS AND KURTOSIS:

In Fig. 6, we have shown the skewness (S) and the kurtosis (B) defined as

$$S = \frac{[\langle \delta q^3 \rangle]}{[\langle \delta q^2 \rangle]^{3/2}}, \quad B = 3.0 - \frac{[\langle \delta q^4 \rangle]}{[\langle \delta q^2 \rangle]^2} \quad \text{with} \quad \delta q = q - \langle q \rangle. \quad (1)$$

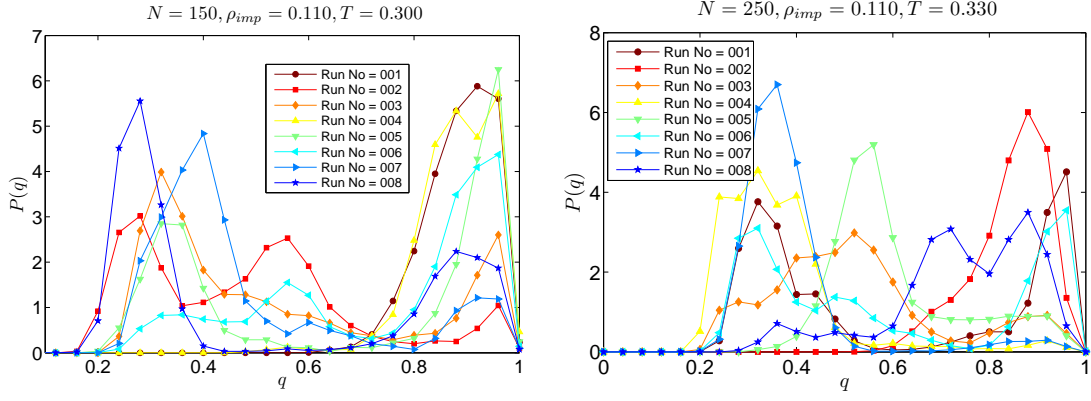


FIG. 5: Left Panel : Distribution of overlap $P(q)$ for $N = 150$ system size with $\rho_{imp} = 0.110$ for the temperature $T = 0.300$ for 8 different samples. Right Panel : Distribution for the $N = 250$ system size at $T = 0.330$.

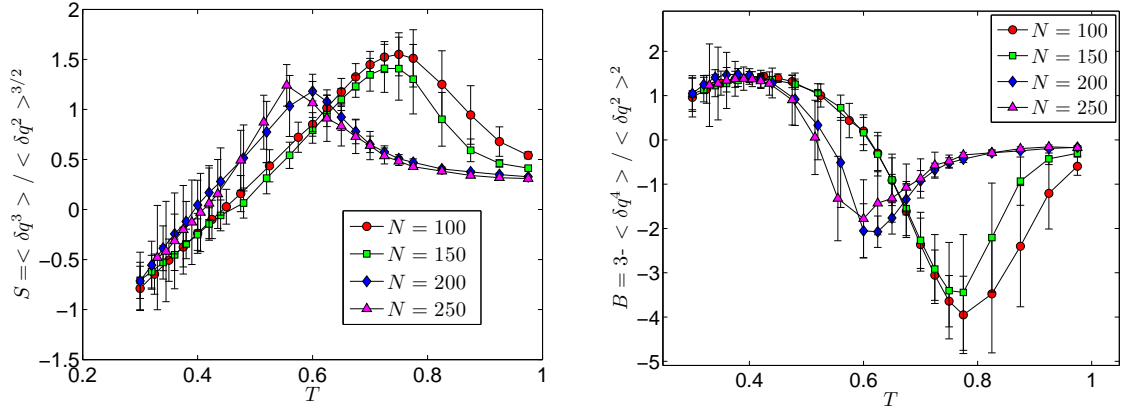


FIG. 6: Left Panel : Temperature dependence of skewness S for different system sizes with $\rho_{imp} = 0.110$. Right Panel : Kurtosis for the same data set.

The usual Binder cumulant is equal to $g(T) = B/2.0$ according to our definition. Notice that $g(T)$ is non-monotonic function of temperature. In conventional phase transition this is very useful quantity which has been used extensively in finite size scaling analysis for determining the critical temperature in the thermodynamic limit. This quantity for different system sizes cross each other at critical temperature because of the finite size scaling.

At zero magnetic field in the Sherrington-Kirkpatrick model for spin glass and the short range Edward-Anderson model $g(T)$ is always positive function of T and increases regularly with decreasing temperature and the skewness is identically zero. Unfortunately, as soon the magnetic field is different from zero already in the Sherrington-Kirkpatrick model (where mean field theory is correct) the behavior at the transition point for the cumulant is much more complex and also for 1024 spins we are very far from the asymptotic limit [2].

The behaviour we see here is distinctly different from this simple case. Moreover it is non-monotonic and mostly a negative function of temperature over the almost whole range. This stems out from the fact that the distribution of q , $P(q)$ starts to become non-Gaussian at quite high temperature. This behaviour is similar to the one seen for the finite range 3-spin model in 4 dimensions [3], where the skewness is strongly decreasing towards the large negative values when the temperature decreases. This similarity once again seems to suggest that this model may also in the same universality class that of short range 3-spin glasses which apparently have their own universality class. We need

further study to clearly find out the exact nature of the transition if any.

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